



C H A P T E R



5

*Effective Strategies for Teaching
Mathematics*



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THIS CHAPTER DESCRIBES considerations for improving instruction for learners at diverse performance levels. Improved instruction alone cannot meet all the challenges that the needs of such learners present. However, the contribution of improved instruction can be enormous and can play a central role in any serious school improvement effort. As noted in Chapter 1, a separate technical report summarizes the research basis for these considerations (Dixon, Carnine, & Kameenui, 1992). While these considerations also contribute to the learning of average and above-average students, the considerations are particularly important for diverse learners. Most of the examples in this chapter are taken from a mathematics program designed to accommodate diverse learners, *Connecting Math Concepts* (Engelmann, Carnine, Kelly, & Engelmann, 1994). The first consideration—organizing content around big ideas—is particularly beneficial for all students, including high-performing students.

The unique curricular needs of diverse learners in meeting high standards were acknowledged by the National Council of Teachers of Mathematics (NCTM) in their publication, *Curriculum and Evaluation Standards for School Mathematics* (1989):

Students with Different Needs and Interests. The consequences of dealing with students with different talents, achievements, and interests have led to such practices as grouping and tracking and to special programs for gifted or handicapped students who need and deserve special attention. However, we believe that *all* students can benefit from an opportunity to study the core curriculum specified in the *Standards*. This can be accomplished by expanding and enriching the curriculum to meet the needs of each individual student, including the gifted and those of lesser capabilities and interests. We challenge teachers and other educators to develop and experiment with course outlines and grouping patterns to present the mathematics in the *Standards* in a meaningful, productive way (p. 253).

The focus of this chapter, then, can be thought of as specific recommendations for meeting the NCTM challenge.

CURRENT ISSUES IN MATHEMATICS INSTRUCTION

There is little, if any, real controversy over the goals of mathematics instruction, as exemplified within the NCTM *Standards* (1989): students should (a) learn to value mathematics, (b) become confident in their ability to do mathematics, (c) become mathematical problem-solvers, (d) learn to communicate mathematically, and (e) learn to reason mathematically. It is important to note that mathematics educators are not recommending watered-down or second-rate mathematics content for diverse learners. Rather, the goal is to devise curriculum and instruction techniques so that these students can think, solve problems, and reason.

Controversy arises when we look specifically at *how* learners in general, and students with learning difficulties in particular, can best achieve the goals

of mathematics instruction. Even then, much of the controversy may be largely a result of relative emphasis.

For example, the NCTM *Standards* (1989) place a strong emphasis on the use of manipulatives in mathematics instruction, based on the assumption that students will subsequently discover an understanding of the algorithms associated with manipulatives activities. However, Evans (1990) found that students mastered concepts regardless of whether work with manipulatives preceded or followed work on algorithms, but that the students who started with algorithms mastered the concepts far more quickly. Given that students with learning difficulties need to “catch up” in some sense, such efficiency is notable. Throughout this chapter, therefore, our emphasis is on the efficiency, as well as the effectiveness, of mathematics instruction.

PRINCIPLES FOR IMPROVING MATH INSTRUCTION STRATEGIES

Designing Instruction Around Big Ideas

Educational tools that are going to facilitate students reaching world-class standard should be organized around big ideas (or fundamental knowledge or root meanings) because these represent major organizing principles, have rich explanatory and predictive power, help frame significant questions, and are applicable in many situations and contexts. For example, in *Factors and Multiples* from the Middle Grades Mathematics Project, the authors write that the program “. . . focuses on this fundamental theorem and related ideas, such as factor, divisor, multiple, common factor, common multiple, relative prime, and composite” (p. 2). The senior author of that book, Glenda Lappan, chaired the NCTM Commission on Teaching Standards for School Mathematics (1991).

Often, however, big ideas in mathematics are not at all clear to students, or even to the teacher. For example, in geometry, students are typically expected to learn seven formulas to calculate the volume of seven three-dimensional figures:

$$\text{Rectangular prism: } l \cdot w \cdot h = v$$

$$\text{Wedge: } \frac{1}{2} \cdot l \cdot w \cdot h = v$$

$$\text{Triangular pyramid: } \frac{1}{6} \cdot l \cdot w \cdot h = v$$

$$\text{Cylinder: } \pi \cdot r^2 \cdot h = v$$

$$\text{Rectangular pyramid: } \frac{1}{3} \cdot l \cdot w \cdot h = v$$

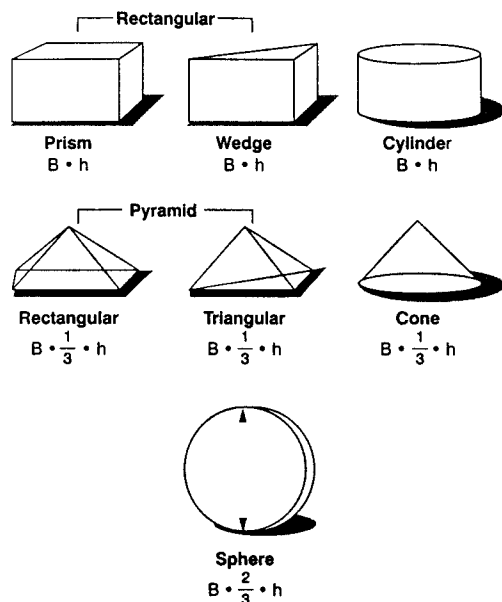
$$\text{Cone: } \frac{1}{3} \cdot \pi \cdot r^2 \cdot h = v$$

$$\text{Sphere: } \frac{4}{3} \cdot \pi \cdot r^3 = v$$

These equations emphasize rote formulas rather than big ideas. An analysis based on big ideas reduces the number of formulas students must learn from seven to slight variations of a single formula—area of the base times the height ($B \cdot h$). This approach enhances understanding while simultaneously reducing the quantity of content to be learned, remembered, and applied. (See Figure 5-1.)

5-1

as the "Big Idea" of
the Base Times a
of the Height



For the regular figures in Figure 5-1—the rectangular prism (box), the wedge, and the cylinder—the volume is the area of the base times the height ($B \cdot h$). For figures that come to a point—the pyramid with a rectangular base, the pyramid with a triangular base, and the cone—the volume is not the area of the base times the height, but the area of the base times $\frac{1}{3}$ of the height ($B \cdot \frac{1}{3} \cdot h$). The sphere is a special case: the area of the base times $\frac{2}{3}$ of the height ($B \cdot \frac{2}{3} \cdot h$), where the base is the area of a circle that passes through the center of the sphere and the height is the diameter.

This analysis of root meaning fosters understanding of the big idea that volume is a function of the area of the base times some multiple of the height. As Gelman (1986) stated, "a focus on different algorithmic instantiations of a set of principles helps teach children that procedures that seem very different on the surface can share the same mathematical underpinning and, hence, root meanings" (p. 350).

Designing Conspicuous Strategies

When students orchestrate multiple concepts in some fashion, they are executing a strategy. Any routine that leads to both the acquisition and utilization of knowledge can be considered a strategy (Prawat, 1989). While the ultimate

purpose of a strategy is meaningful application, acquisition is most reliable for diverse learners when initial instruction explicitly focuses on the strategy itself, rather than its meaningful application.

Consider, for example, the following problem, presented to each fifth-grade class in a school:

At lunch, each student can choose a carton of white or chocolate milk. Estimate how many cartons of chocolate and white milk should be ordered for the entire school.

For students to work such problems successfully, they must have both computational ability and well developed strategies for data gathering, proportions, and probability which are relevant to a broad range of real mathematical problems.

In contrast to such well developed strategies, strategies may be so specific and narrow in application that they are little more than a rote sequence for solving a particular problem or a very small set of highly similar problems. For example, in a study by McDaniel and Schlager (1990) on water-jar problem solving, students in one of the teaching conditions learned a rote formula for adding and removing amounts of water with different-sized jars (+1 -2 +1), which, predictably, did not transfer well to solving other water-jar problems.

Too often, mathematics knowledge appears to be rote. Davis (1990) points out that "traditional school practice" tests mainly the ability to repeat back what has been told or demonstrated. There are really two significant problems with the "traditional school practice" described by Davis. First, such practice is often directed toward "small ideas"—for example, arbitrary procedures such as cross-multiplying to solve problems like $x/a = b/c$. Second, such procedures are frequently "repeated back" for rote recall, effectively preempting the possibility that students will even infer the important mathematical principles underlying them.

At the other extreme, a "strategy" may be so general that it is little more than a broad set of guidelines. Such strategies may be better than nothing, but they do not dependably lead most students to solutions for most problems. For instance, a broad strategy such as "draw a picture" or "read, analyze, plan, and solve" is probably far too general for reliably leading a majority of students to reasonable solutions for complex problems such as the milk-ordering problem above.

An important goal for strategy instruction is that the strategies taught are "good" in some sense. Some students develop strategies that are too narrow or too broad, while others develop strategies that are "just right." A major challenge of instruction—perhaps the major challenge—is to develop "just right" strategies for interventions with those students who do not develop strategies on their own, including, but not limited to, diverse learners.

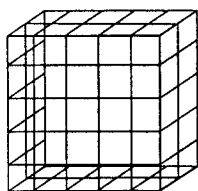
Based on an exhaustive review of research, Prawat (1989) recommends that efficient strategy interventions should be "intermediate in generality." That is, efficient strategies fall somewhere between the extremes of being narrow in application but, presumably, relatively easy to teach successfully, and being broad but not necessarily reliable or easy to teach. This suggests that the principal feature of a "good strategy" is that it adheres to the Law of Parsimony

as it applies to evaluating competing theories: "That theory is best that explains the most in the simplest way" (Mouly, 1978). As applied specifically to evaluating strategies, the Law of Parsimony might read: That strategy is best that results in the greatest number of students from a targeted student population (such as diverse learners) being able to successfully solve the greatest number of problems or complete the broadest range of tasks by applying the fewest possible strategic steps.

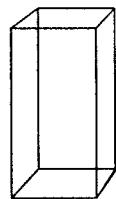
When experts implement strategies to acquire and utilize knowledge, only the result is overt; the steps in the strategy the experts follow are covert. The whole purpose of developing instructional strategies is to explicate expert cognitive processes so that they become visible to non-expert learners. The research support for explicitly teaching conspicuous strategies is quite strong (Carnine & Stein, 1981; Charles, 1980; Gleason, Carnine, & Boriero, 1990; Leinhardt, 1987; Resnick, Cauzinille-Marmeche, & Mathieu, 1987; Resnick & Omanson, 1987).

An example of a conspicuous strategy for the volume formula appears below. Note that the first step prompts the connection with a more concrete representation of volume in which students can count the cubes in a figure. Step 2 introduces the strategy. In step 3, the teacher does not assist the students because they have already been taught to compute the area of a rectangle. In contrast, step 4 calls for a new calculation, so the teacher is more directive.

1. *Linkage to prior knowledge:* "Touch box a. You know how to figure out its volume. Count the cubes and write the volume. What did you write? Yes, 50 cubic meters."
2. *Introduction of new strategy:* "Touch box b. You're going to learn how to calculate the volume by multiplying the area of the base times the height."
3. *Computing the area of the base:* "First calculate the area of the base for box b."
4. *Computing the volume:* "To figure out the volume of the box, you multiply the area of the base times the height. What are the two numbers you will multiply? Yes, 6×7 ."
5. *Writing the complete answer:* Write the answer with the appropriate unit. What did you write? Yes, 42 cubic inches.
 - a. Count the cubes:
 - b. Multiply the area of the base times the height:



Meters



3 inches

7 inches

2 inches

The applicability of the big idea for volume with variations of a single strategy for three-dimensional figures is obvious. In contrast, it is not at all obvious how a single big idea with variations of a strategy could link these six problems:

1. Five packages of punch mix make 4 gallons. How many gallons of punch can Juan make for the party with 15 packages?
2. How long will it take a train to go 480 miles to Paris if it travels at 120 mph?
3. What is the average rate of a car that goes 450 miles in 9 hours?
4. How many pounds is 8 kilograms?
5. The oil transferred from the storage area has filled 44 tanks. There are 50 tanks. What percentage of the tanks are full?
6. There are 52 cards in a deck. Thirteen of them are hearts. The rest are not hearts. If you took trials (drew a card and then replaced it) until you drew 26 hearts, about how many trials would you expect to take?

However, it is with such seemingly unrelated problem types that a strategy based on a big idea is most valuable, particularly with learners for whom such connections usually remain elusive. The big idea that connects these different problems types is proportions. The strategy for proportions must be applied to each problem type in a systematic manner, to make clear that the same big idea underlies these very different problems. The application of proportions is most obvious in the first problem:

Five packages of punch mix make 4 gallons. How many gallons of punch can Juan make with fifteen packages?

A medium-level strategy for proportions might first have students map the units:

$$\frac{\text{packages}}{\text{gallons}}$$

Next, students insert the relevant information:

$$\frac{\text{packages}}{\text{gallons}} \quad \frac{5}{4} = \frac{15}{\square}$$

Finally, students solve for the missing quantity:

12 gallons

Rate problems, which are not typically viewed as proportion problems, also can be solved through a proportion strategy. Note that the key to setting up rate problems as proportions is realizing that the ratio in the proportion is a number of distance units over a single unit of time. This principle is applicable to solving the second problem:

How long will it take a train to go 480 miles to Paris if it travels at 120 mph?

First, map the units. The abbreviation mph can be represented as:

$$\frac{\text{miles}}{\text{hour}}$$

Next, insert the relevant information:

$$\frac{\text{miles}}{1} \frac{120}{1} = \frac{480}{\square}$$

Finally, solve for the answer:

4 hours

In the next rate problem, students solve for the average rate:

What is the average rate of a car that goes 450 miles in 9 hours?

Map the units:

$$\frac{\text{miles}}{\text{hour}}$$

Insert the relevant information:

$$\frac{\text{miles}}{\text{hour}} \frac{\square}{1} = \frac{450}{9}$$

Solve for the answer:

50 miles per hour

Another application of proportions occurs with measurement equivalences. The key to this problem type is that students set up a ratio between the two units involved in the equivalence.

How many pounds is 8 kilograms?

Map the units:

$$\frac{\text{pounds}}{\text{kilograms}}$$

Insert the relevant information:

$$\frac{\text{pounds}}{\text{kilograms}} \frac{2.2}{1} = \frac{\square}{8}$$

Solve for the answer:

17.6 pounds

Similarly, percent problems can be set up as proportions. For percents, the key to treating them as proportions is labeling the second ratio as telling about the percentage and pointing out that the denominator of the percentage ratio, which is almost always unstated, is 100.

The oil transferred from the storage area has filled 44 tanks. There are 50 tanks. What percentage of the tanks are full?

Map the units:

$$\frac{\text{filled tanks}}{\text{total tanks}}$$

Insert the relevant information:

$$\frac{\text{filled tanks}}{\text{total tanks}} \frac{44}{55} = \frac{\text{percent}}{100}$$

Solve for the answer:

88 percent

With the following, more difficult percentage problem, the proportion strategy makes the problem quite manageable, even for students with learning difficulties.

The oil transferred from the storage area filled 44 tanks. So far, 88% of the oil in the storage area has been transferred into tanks. How many tanks will be filled when all the oil is transferred from the storage area?

Map the units:

$$\frac{\text{filled tanks}}{\text{total tanks}}$$

Insert the relevant information:

$$\frac{\text{filled tanks}}{\text{total tanks}} \frac{44}{\square} = \frac{\text{percent}}{100}$$

Solve for the answer:

50 tanks

The next problem type, illustrating odds and probability, also has a key for tying it to proportions: Setting up a ratio of one type of member to another type or to the total number of members. In the example that follows, the one type of winning trials is related to the total trials.

There are 52 cards in a deck. Thirteen of them are hearts. The rest are not hearts. If you took trials (drew a card and then replaced it) until you drew 26 hearts, about how many trials would you expect to take?

Map the units:

$$\frac{\text{hearts}}{\text{trials}}$$

Insert the relevant information:

$$\frac{\text{hearts}}{\text{trials}} = \frac{13}{52} = \frac{26}{\square}$$

Solve for the answer:

104 trials

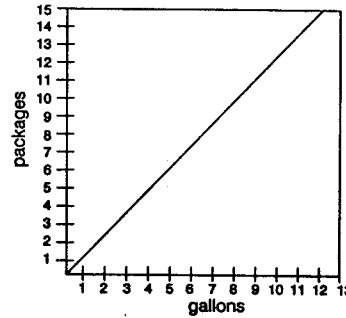
The next connection to be illustrated with proportions involves the coordinate system. Proportions can link simple proportion problems—rate, measurement equivalence, percentage, and probability—to the coordinate system. This linkage is illustrated in the graphs for each problem type in Figure 5-2.

The last connection—functions—is also apparent in Figure 5-2. A function table accompanies each graph in the figure.

The application of the proportion big idea with variations of a strategy for these problem contexts will deepen the student's understanding not only of proportions but also of rate, measurement equivalencies, percentage, probability, the coordinate system, and functions. One of the most important ways to develop this understanding is through learning how various concepts are linked by a single strategy. In other words, a deep understanding of proportions is constructed by applying the strategy across many contexts. For this reason, the application of a strategy can be thought of as more important in developing understanding and proficiency than how the meaning of a strategy is initially constructed. These applications do far more to develop deep understanding than allowing students to initially construct their own meaning for proportions in authentic activities. Becoming proficient at authentic activities is far more important than starting out with authentic activities.

Five packages of punch mix make 4 gallons. Fifteen packages would make how many gallons of punch?

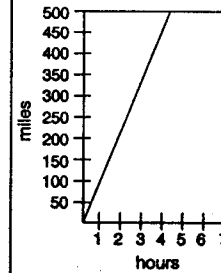
$$\frac{\text{packages}}{\text{gallons}} = \frac{5}{4} = \frac{15}{\square}$$



| packages | p | 1 | 2 | 3 | 4 | 5 |
|----------|--------|----|-----|-----|-----|-----|
| gallons | .8 x p | .8 | 1.6 | 2.4 | 3.2 | 4.0 |

How long will it take a train to go 480 miles to Rome if it travels at 120 mph?

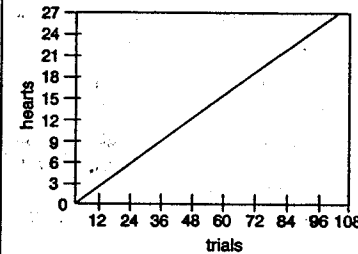
$$\frac{\text{miles}}{\text{hour}} = \frac{120}{1} = \frac{480}{\square}$$



| miles | 120 x h | 120 | 240 | 360 | 480 |
|-------|---------|-----|-----|-----|-----|
| hour | h | 1 | 2 | 3 | 4 |

There are 52 cards in a deck. 13 of these are hearts. The rest are not hearts. If you took trials (drew a card and then replaced it) until you drew 26 hearts, about how many trials would you expect to take?

$$\frac{\text{hearts}}{\text{trials}} = \frac{13}{52} = \frac{26}{\square}$$



| hearts | h | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
|--------|-------|----|----|----|----|----|----|----|----|-----|
| trials | 4 x h | 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | 108 |

FIGURE 5-2 Using Proportions to Link Multiple Concepts to the Coordinate System

Designing Mediated Scaffolding

British educator A. J. Romiszowski has characterized traditional mathematics instruction as: "I'll work two on the board, then you do the rest." The "I'll work two" part of that approach can be thought of as a model, and the "you do the rest" is considered immediate testing. It has been said that the problem with learning from experience is that the lessons come too late. The same could be said of this traditional model of instruction. After "doing the rest," students might receive feedback, ranging from right/wrong, to an explanation of how to do missed problems, and possibly a grade. The feedback is too late and the grade too early.

Scaffolding is a means by which students receive support in various forms along the path to full understanding and "doing the rest" successfully. Along the way, teachers would remove more bits of scaffolding, but in no instance would they abruptly remove all the scaffolding and, in essence say, "You do the rest." For example, after modeling the formula the teacher would not have the students "do the rest." Instead, the teacher would scaffold the steps of the strategy, initially giving students feedback after every step. The steps for a scaffolded volume strategy involving a cone 5 inches tall with a radius of 1.6 inches might take this form:

1. "Write the formula for the volume of the figure."

Students write: $B \cdot \frac{1}{3} \cdot h$

2. "Calculate the area of the base for that figure."

Students write: $3.14(1.6)^2 = 8.04$

3. "Calculate the volume."

Students write: $8.04 \cdot \frac{1}{3} \cdot 5 = 13.4$

4. "Write the complete answer with the appropriate unit."

Students write: 13.4 cubic inches.

This level of scaffolding is specific enough to be useful but general enough to be used flexibly with all three-dimensional figures. One test for flexibility is the degree to which the strategy can be applied to seemingly different problem types, as was illustrated earlier with proportions. This flexibility is also the means by which students come to understand how various big ideas can be linked, such as proportions, functions, and the coordinate system.

Designing Primed Background Knowledge

In all of the proportion examples, a strategy is applied to a variety of concepts. The concepts to which the proportions strategy is applied—rate, percentage, measurement, probability—are assumed to have been introduced previously. Similarly, when proportions are linked to other big ideas—coordinate system and functions—these big ideas would need to have been taught. Without such prior knowledge, the application of proportions could be a rote activity, extending stu-

dents' understanding of neither the proportions strategy, nor the concepts to which the strategy is applied, nor the other big ideas to which the strategy is linked. Similarly, the strategy for volume assumes certain prior knowledge on the part of the students: an understanding of the concept of area as well as computational proficiency, e.g., squaring a number and then multiplying by π .

Providing students with both the necessary prior knowledge and flexible strategies based on big ideas that can link that knowledge is possibly the best way to prepare diverse learners for the challenges posed by the new NCTM *Standards*. Instruction should purposefully demonstrate a broad range of mathematics applications for students and enable them to successfully engage in such applications by providing the necessary prior knowledge.

Designing Strategic Integration

Students must not only understand important mathematics strategies as entities, but must also learn the relationships among strategies leading to an integrated, cohesive strategy (Nickerson, 1985; Prawat, 1989; Van Patten, Chao, & Reigeluth, 1986). It is conceivable that a student could learn several "good" strategies but not know when to apply them. However, instruction on individual strategies can be designed to anticipate situations in which several strategies are integrated, a practice also consistent with Piaget's (1973) model of assimilation of the new to the old and accommodation of the old to the new.

For example, if diverse learners are going to have opportunities to successfully engage in solving novel problems, they must not only be able to apply a strategy such as that for proportions but also be able to know when *not* to apply the particular strategy. Understanding involves knowing when a strategy applies and when it does not. Developing such understanding in diverse learners requires integrated teaching, not in the broad sense of interdisciplinary teaching, but within a discipline. Teaching for integration within mathematics can be illustrated with the proportion strategy. Problem A is a fairly straightforward proportion problem:

Problem A. A truck delivers cartons of juice to a store. $\frac{2}{7}$ of the juice is grape. The truck has 8400 cartons of juice. How many are grape juice?

Map the units: $\frac{\text{Grape}}{\text{Total}}$

| | <i>Ratio</i> | <i>Juice Cartons</i> |
|----------------------------------|---|---|
| Insert the relevant information: | $\frac{\text{Grape}}{\text{Total}} = \frac{2}{7}$ | $= \frac{\boxed{}}{8400}$ |

Solve for the answer: 2400 cartons

If students erroneously apply the same proportion strategy to the numbers in problem B, the answer will also be 2400.

Problem B: A truck delivers cartons of grape and apple juice to a store. $\frac{2}{7}$ of the juice is grape. The truck will deliver 8400 cartons of apple juice. How many cartons of grape juice will the truck deliver?

Map the units: $\frac{\text{Grape}}{\text{Total}}$

Insert the relevant information: $\frac{\text{Ratio}}{\text{Total}} = \frac{\text{Juice Cartons}}{\text{8400}}$

$$\frac{2}{7} = \frac{\boxed{2400}}{8400}$$

Solve for the answer: 2400 cartons

For Problem B, the answer 2400 is *incorrect*, of course, because the 8400 does not refer to the total number of cartons, but to apple juice cartons.

With integrated teaching, the students would be less likely to inappropriately apply the basic proportion strategy to problem B because an advanced proportion strategy would have been taught to “accommodate” these more complex problem types that deal with not just two elements (total juice and grape juice), but with three elements (total juice, grape juice, and apple juice). Conversely, the new strategy for three elements must be “assimilated” with the simpler strategy that handles only two elements.

This accommodation and assimilation is accomplished through an advanced proportion strategy for three elements. This strategy illustrated in Figure 5-3.

The advanced proportion strategy in Figure 5-3 is in itself a medium-level strategy that can be flexibly applied to more complex mathematics problems, such as those involving mixtures and discounts. Below is the map for the application of the advanced proportion strategy to a discount problem:

A shirt was on sale. The discount was \$2. The sale price was \$18. What percent was the discount?

| | Dollars | Percent |
|----------------|----------------------|----------------------|
| Sale Price | 18 | <input type="text"/> |
| Discount | 2 | <input type="text"/> |
| Original Price | <input type="text"/> | 100 |

The application of the complete strategy to a mixture problem is illustrated in Figure 5-4.

Designing Judicious Review

The term “review” can be an emotive one in education, conjuring up images of endless (and, perhaps, mindless) drill and practice. Yet research strongly sup-

FIGURE 5-3
Advanced Proportion Strategy

A truck delivers cartons of grape and apple juice to a store. $\frac{2}{7}$ of the juice is grape. The truck will deliver 8400 cartons of apple juice. How many cartons of grape juice will the truck deliver?

| | | |
|---|-------|---------------------------------------|
| Step 1: The students map 3 units, not just 2, and insert the relevant information: | Ratio | Juice Cartons |
| | Grape | 2 <input type="text"/> |
| | Apple | <input type="text"/> 8400 |
| Total | | 7 <input type="text"/> |
| Step 2: The students use their knowledge of missing addends to come up with the unknown value in the ratio column: $7 - 2 = \boxed{5}$ | Grape | 2 <input type="text"/> |
| | Apple | <input type="text"/> 8400 |
| | Total | 7 <input type="text"/> |
| Step 3: The students write and solve the proportion to determine the number of cartons of grape juice: $\frac{2}{5} = \frac{\boxed{3360}}{8400}$ | Grape | 2 <input type="text"/> $\boxed{3360}$ |
| | Apple | <input type="text"/> 8400 |
| | Total | 7 <input type="text"/> |

ports certain review practices as significantly effective. We include review as the last guideline because in many ways effective review depends on the extent to which other considerations are reflected in instructional tools.

It can be said that one gets out of review what one puts into it; that is, the quality of instruction—principally in terms of big ideas and strategies—influences the value of review. Regardless of how much review is devoted to “small ideas” or marginally significant material, the ideas remain small and the material marginally significant.

The following are considerations for effective review (Dempster, 1991):

1. *Review should be sufficient.* Is there enough review to achieve the goals of fluency and understanding? For example, solving a wide variety of proportion problems depends on fluency and understanding of computation with proportions, even if carried out with a calculator. If diverse learners, in particular, are to use proportions to solve problems, they will need to practice and review computation with many, many proportion problems.

FIGURE 5-4
Application of Advanced Proportion Strategy Applied to a Mixture Problem

Mixture Problem

A mix contains peanuts and almonds in a ratio of 4 to 3. If 35 pounds of mix are made, how many pounds of almonds will be used?

| | | |
|---|--|---------------------------------|
| <p>Step 1: The students map the units and write the known values:</p> | <i>Ratio</i> | <i>Pounds</i> |
| | Peanuts 4 | <input type="text"/> |
| | Almonds 3 | <input type="text"/> |
| | Total <input type="text"/> | 35 |
| <p>Step 2: The students use their knowledge of missing addends to come up with the unknown value in the ratio column: $4 + 3 = \boxed{7}$</p> | Peanuts 4 | <input type="text"/> |
| | Almonds 3 | <input type="text"/> |
| | Total <input type="text"/> | 35 |
| <p>Step 3: The students write and solve the proportion to determine the number of pounds of almonds: $\frac{3}{7} = \frac{\boxed{15}}{35}$</p> | Peanuts 4 | <input type="text" value="20"/> |
| | Almonds <input type="text" value="3"/> | <input type="text" value="15"/> |
| | Total 7 | 35 |

2. *Review should be distributed.* Given a fixed number of review opportunities, that number will enhance learning better if it is distributed over time than if it is massed. Specifically, distributed review contributes to long-term retention and automaticity of knowledge. The distribution of review suggests that the many, many proportion problems that students might work should not all be crammed into a few days. Several problems should be reviewed over many days.

3. *Review should be cumulative.* This requirement is tied closely to the integration guideline. The notion of cumulative review means that material taught accumulates in review. For example, after learning to use a multiplication and division strategy to solve problems, and after learning a proportion strategy to solve problems, problem-solving exercises should consist of a mix, some calling for multiplication/division and some calling for proportions.

4. *Review should be varied.* With relatively few exceptions, the specific items that are reviewed should not be the same as the items used earlier in instruction. The reason for this is that varied items promote generalization

and transference. However, items should not be so varied that they actually represent new knowledge. For example, instruction on using proportions to calculate measurement equivalencies might focus on units of weight and length. Review exercises might include capacity, with standard units as well as metric units. However, review problems should not include conversions of Celsius to Fahrenheit because mapping that conversion goes beyond the use of the proportion strategy. New, difficult problem types such as this conversion merit instruction, rather than just inclusion in a review activity.

SUMMARY

The tool design considerations for mathematics presented in this chapter have significance for those working directly with students and for those developing and publishing mathematics instructional materials. Vygotsky (1978) uses the term *zone of proximal development* to describe situations in which students' cognitive ability matches the cognitive requirements demanded by an instructional activity. The importance of designing educational strategies to match students' zone of proximal development is critical to ensuring that students benefit from instruction and that the instructional experiences enhance the students' self-esteem.

In solving the milk-ordering problem described in the section on conspicuous strategies, students integrate the advanced proportion strategy with data gathering and probability-statistics strategies. (See Figure 5-5.) The students apply data gathering strategies—determining the ratio of chocolate milk to white milk for their class and finding out the total enrollment of the school. The students then evoke the advanced proportions strategy: mapping the relevant information. The fifth-graders would need to assume that the preference for types of milk in their class represents the whole school's preference, which entails applying the concept of sampling from statistics and probability. Finally, the concept of missing addends is evoked to solve for the number for white milk.

Many subtle variations are possible with a problem like this, but all are accommodated through the integrated strategies illustrated in Figure 5-5. One variation might be to use average attendance instead of total enrollment, which would cut down on milk ordered (and wasted). In another variation, students might have reason to believe that the preferences in their own class would not be representative of the entire school. The students could gather data from different classes, work the problem, compare the results, and discuss variations in solutions based on different samples. As a final variation, the students could compare results with actual figures on milk ordered to predict shortages or excesses of each type of milk.

The milk-ordering problem and its variations illustrate how goals of the NCTM—working together to enhance understanding, engage in conjecture and invention, and connect mathematical ideas—can be effectively met for diverse learners through judicious instruction in medium-level strategies based on big ideas. As students discuss their options for selecting a sample group, they are working together to enhance their understanding of mathematics. As they

FIGURE 5-5
Data Gathering, Advanced Proportions, and Probability-Statistics Strategies

| <p>Step 1: Data Gathering The students conduct a survey in their class to determine the preferences for white and chocolate milk. The students also find out from the office the total enrollment for the school.</p> | <p>There are 32 students in the class; 22 prefer chocolate milk and the rest prefer white.</p> <p>There are 479 students in the school.</p> | | | | | | | | | | | | |
|--|---|------------------|----------------------|------------------|-----------|----|-----|-------|----|-----|-------|----|-----|
| <p>Step 2: Advanced Proportions The students map the units for the advanced proportions strategy and insert the relevant information.</p> | <table border="1"> <thead> <tr> <th></th> <th style="text-align: center;">Fifth-grade Class</th> <th style="text-align: center;">Entire school</th> </tr> </thead> <tbody> <tr> <td>Chocolate</td> <td style="text-align: center;">22</td> <td style="text-align: center;">□</td> </tr> <tr> <td>White</td> <td style="text-align: center;">10</td> <td style="text-align: center;">□</td> </tr> <tr> <td>Total</td> <td style="text-align: center;">32</td> <td style="text-align: center;">479</td> </tr> </tbody> </table> | | Fifth-grade Class | Entire school | Chocolate | 22 | □ | White | 10 | □ | Total | 32 | 479 |
| | Fifth-grade Class | Entire school | | | | | | | | | | | |
| Chocolate | 22 | □ | | | | | | | | | | | |
| White | 10 | □ | | | | | | | | | | | |
| Total | 32 | 479 | | | | | | | | | | | |
| <p>Step 3: Probability and Statistics The students solve a proportion to estimate the number of chocolate milk cartons to purchase for the entire school:</p> $\frac{22}{32} = \frac{\boxed{329}}{479}$ | <table border="1"> <tbody> <tr> <td>Chocolate</td> <td style="text-align: center;">22</td> <td style="text-align: center;">329</td> </tr> <tr> <td>White</td> <td style="text-align: center;">10</td> <td style="text-align: center;">□</td> </tr> <tr> <td>Total</td> <td style="text-align: center;">32</td> <td style="text-align: center;">479</td> </tr> </tbody> </table> | Chocolate | 22 | 329 | White | 10 | □ | Total | 32 | 479 | | | |
| Chocolate | 22 | 329 | | | | | | | | | | | |
| White | 10 | □ | | | | | | | | | | | |
| Total | 32 | 479 | | | | | | | | | | | |
| <p>Step 4: Missing Addends The students determine the estimate for white milk using their knowledge of missing addends: $479 - 329 = 150$</p> | <table border="1"> <tbody> <tr> <td>Chocolate</td> <td style="text-align: center;">22</td> <td style="text-align: center;">329</td> </tr> <tr> <td>White</td> <td style="text-align: center;">10</td> <td style="text-align: center;">150</td> </tr> <tr> <td>Total</td> <td style="text-align: center;">32</td> <td style="text-align: center;">479</td> </tr> </tbody> </table> | Chocolate | 22 | 329 | White | 10 | 150 | Total | 32 | 479 | | | |
| Chocolate | 22 | 329 | | | | | | | | | | | |
| White | 10 | 150 | | | | | | | | | | | |
| Total | 32 | 479 | | | | | | | | | | | |

weigh the relative merits of using total enrollment versus average attendance, they are engaging in conjecture and invention. As they link their understandings of various strategies, they are clearly learning to connect mathematical ideas, solve problems, and apply mathematics broadly.

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