

## Teaching Problem-Solving Strategies for Word Problems to Students with Learning Disabilities

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Studies of students with learning disabilities (LD) disclose extensive underachievement in mathematics, including problem-solving. Cawley and Miller (1989) reported that 8- and 9-year-olds with LD performed at about the first-grade level, while 16- and 17-year-old students with LD scored at about the fifth-grade level. Algozzine, O'Shea, Crews, and Stoddard (1987) observed that students with disabilities mastered "very few mathematical skills" to a high degree compared to their tenth-grade peers; performance on mathematics analysis or problem-solving was particularly poor. Barron, Bransford, Kulewicz, and Hasselbring (1989) reported findings consistent with Algozzine et al. (1987) and speculated that students with disabilities have failed to automatize even the most basic skills that would allow them to concentrate on problem-solving. As a consequence, students with LD and their teachers are caught between low levels of achievement and increasing expectations for achievement.

The NCTM Curriculum and Evaluation Standards (1989) articulate the expectations for achievement this way: "As children progress through the grades, they should encounter more diverse and complex types of problems that arise from both real-world and mathematical contexts. When problem-solving becomes an integral part of classroom instruction and children experience success in solving problems, they gain confidence in doing mathematics and develop persevering and inquiring minds. They also grow in their ability to communicate mathematically and use higher-level thinking processes" (p. 23). This discrepancy between expectation and achievement levels is particularly great in the area of word problems, probably because strategies for solving these problems entail competence not only in mathematics but also in language. High-achieving students are those most likely to infer useful strategies from their experience in the least amount of time, whereas students with LD and inexperienced students tend not to use effective or appropriate strategies to solve problems. Unless students receive systematic instruction in the use of strategies (e.g., Montague, 1992; Montague & Bos, 1986, 1990), they often use inefficient strategies or use strategies ineffectively.

The "key word" strategy is an example of a potentially ineffective strategy that many students with LD arrive at by themselves or, in some cases, are taught. In using this strategy, students simply associate key words (such as "more," "all together," "gave away," "left over") with mathematical operations. Sometimes this strategy works (e.g., "Juan had 9 cookies. He gave away 4. How many did he end up with?" "Gave away" can here be correctly interpreted as indicating subtraction). But many times this strategy does not work (e.g., "Juan gave away 9 cookies in the morning. He gave away 4 cookies in the afternoon. How many cookies did he give away that day?" In this problem, "gave away" cannot be correctly interpreted as indicating subtraction). As these examples illustrate, key word strategies are too narrow to facilitate the correct solution of many math word problems. For example, Carpenter, Hiebert, and Moser (1981) reported that students rarely solved word problems correctly if "key words" appeared in the problems but were irrelevant to their solution.

Narrow strategies, such as the key word strategy, are frequently ineffective because they can mislead students who rely on them. On the other hand, a strategy can be so broad that it does not work for the majority of students most of the time. For example,

although simply drawing a picture to illustrate what is described in a word problem can help some students solve some problems, that approach is too broad to qualify as a reliable strategy for most students with LD.

“Medium” strategies are the most likely to benefit students (Prawat, 1989). The purpose of strategy instruction is to identify and teach those medium strategies, often rules and approaches to problem-solving, that will help students with LD solve complex problems as efficiently as their higher achieving peers—explicitly taught strategies that can let the majority of students in on the “secrets to success.”

A key element of explicit strategies is teaching students how to work from different types of problem maps. These maps are very useful because they teach students not to rely on key words and go directly from a word problem to computation, but first, to represent the information in a map and then work from the map to set up the computation. It is possible to provide students with an increasing repertoire of problem-solving strategies that are fine-tuned to the types of problems they are required to solve. For example, outside of school, mathematical problems usually do not come with neat, verbal clues, and they may deal with comparisons of quantities or deal with quantities in different classifications (see Silbert, Carnine, & Stein, 1990). We will illustrate several mapping strategies appropriate for the elementary and middle grades: diagrams, ratio equations, tables, and inverse-operation equations. The strategies are excerpted from the series *Connecting Math Concepts* (Engelmann, Carnine, Kelly, & Engelmann, 1991-1995).

## DIAGRAMS

All single-operation word problems (addition, subtraction, multiplication, or division) may be diagrammed to identify the operation required to solve the problem. An essential component of the diagram strategy is the concept of a number family (see Scott and Raborn in this series for examples of other types of diagrams).

### Addition/Subtraction

For any addition/subtraction situation, there are two “small” numbers and a “big” number (the sum).

An addition/subtraction number family is mapped this way:

$$\begin{array}{ccc} 7 & 9 & \\ \hline & \longrightarrow & 16 \end{array}$$

The family above represents the following addition/subtraction facts:

$$\begin{array}{ll} 7 + 9 = 16 & 16 - 9 = 7 \\ 9 + 7 = 16 & 16 - 7 = 9 \end{array}$$

A missing big number implies addition:

A missing small number implies subtraction:

$$\begin{array}{l} \square \xrightarrow{22} 30 \\ \text{or } 8 \xrightarrow{\square} 30 \end{array} \quad \begin{array}{l} 30 - 22 = \square \\ 30 - 8 = \square \end{array}$$

Students learn to map a variety of addition/subtraction word-problem types as number families. We will illustrate using two types: comparison problems and classification problems.

**Comparison problems.** In comparison problems, the difference between two values being compared may be information given in a problem (e.g., Marco sold 57 fewer subscriptions than Lui) or the unknown in a problem (e.g., How much heavier was Mary?). Because of the words “sold fewer” in the following problem, many students with LD will subtract.

Marco sold 57 fewer magazine subscriptions than Lui. Marco sold 112 subscriptions. How many subscriptions did Lui sell?

Students can use number families to avoid this confusion. The first step is to represent the problem using a number family; students must determine whether each of the two numbers given in the problem is a small number or the big number. The students are shown a simple way to do this: they find the sentence that tells about the comparison and read it without the difference number. For example, students are taught to read the first sentence without the 57: “Marco sold fewer subscriptions than Lui.” Because Marco sold fewer subscriptions, Marco is represented by a small number. By default, Lui is the big number. The students write M for Marco and L for Lui:

$$\xrightarrow{M} L$$

The word problem also gives a number for the difference between Marco and Lui. That number always has to be a small number. Marco sold 57 fewer, so 57 is the other small number:

$$\xrightarrow{57 \quad M} L$$

Next the students read the rest of the problem. The problem asks about Lui and gives a number for Marco, so the students draw a box around L and replace the M with 112:

$$\xrightarrow{57 \quad \cancel{M} \quad 112} \boxed{L}$$

Because the problem gives both small numbers, the students write an addition problem.

$$\begin{array}{r} 57 \\ + 112 \\ \hline \end{array}$$

The answer tells how many magazine subscriptions Lui sold.

**Classification problems.** To work classification problems, students must understand the relationships among the classes named in a problem. For example, cats and dogs are members of the larger class, pets; or as in the following problem, magazine subscriptions and newspaper subscriptions are members of the inclusive (larger) class, subscriptions. As in the earlier example, a word in the problem is likely to mislead many

students with LD. Rather than adding the numbers (as the word “get” may imply), the students must subtract to solve the problem.

Maria has to get 112 magazine and newspaper subscriptions. She is sure she can get 57 magazine subscriptions. How many newspaper subscriptions does she have to get?

To represent this situation with a number family, students treat subscriptions, the big class, as the big number. The names for the smaller or subordinate classes go in the places for the small numbers, on top of the number family arrow:

magazine   newspaper    $\rightarrow$    subscriptions

The students cross out the words that have number values and draw a box around the word the problem asks about:

~~57~~  
~~magazine~~   newspaper    $\rightarrow$    ~~112~~  
~~subscriptions~~

The students know the big number is 112; so they subtract.

$$112 - 57 = \_$$

The answer is the number of newspaper subscriptions.

### Multiplication and Division

Number families also may be applied to any multiplication/division situation. The two “small” numbers are factors, while the “big” number is the product.

A multiplication/division number family is mapped in this way:

$$\begin{array}{r} 5 \\ 3 \overline{) 15} \end{array} \rightarrow$$

The family above represents the following multiplication/division facts:

$$\begin{array}{ll} 3 \times 5 = 15 & 15 \div 3 = 5 \\ 5 \times 3 = 15 & 15 \div 5 = 3 \end{array}$$

For any multiplication/division number family, a missing *big* number implies multiplication:

$$2 \overline{) \square} \xrightarrow{14} \quad 2 \times 14 = \square$$

A missing *small* number implies division:

$$2 \overline{) \square} \xrightarrow{28} \quad 28 \div 2 = \square$$

or

$$\square \overline{) 28} \xrightarrow{14} \quad 28 \div 14 = \square$$

Students learn to map the values in a multiplication/division word problem (one that deals with equal-sized groups) as a number family. The following word problems describe groups of bottles in cartons:

- Each carton holds 8 bottles. How many cartons are needed for 24 bottles?
- Each carton holds 8 bottles. There are 24 cartons. How many bottles are there?

There are 8 bottles for every carton, so there are more bottles than cartons. Thus the total number of bottles is the big number in the map:

$$\overline{) B} \xrightarrow{\quad}$$

The number of cartons is a small number:

$$\overline{) B} \xrightarrow{C}$$

and the number of bottles in just one carton is the other small number:

$$8 \overline{) B} \xrightarrow{C}$$

We can now substitute the value we know. In the first example there are 24 bottles:

$$8 \overline{) 24} \xrightarrow{C}$$

So the number of cartons is  $24 \div 8$ . The answer is 3 cartons.

In the second example, there are 24 cartons:

$$8 \overline{) 24} \xrightarrow{C}$$

So the number of bottles is  $8 \times 24$ . The answer is 192 bottles.

By mapping the situation before choosing the operation, students apply a strategy that helps them analyze the situation and reduces the likelihood of error.

## RATIO EQUATIONS

Students learn that multiplication and division situations also may be solved with a ratio and proportion strategy. For example:

There are 8 bottles in each carton.  
How many bottles are in 24 cartons?

Students map the first sentence as a ratio:

$$\frac{\text{bottles}}{\text{cartons}} \quad \frac{8}{1}$$

The question the problem asks becomes the second ratio in the proportion:

$$\frac{\text{bottles}}{\text{cartons}} \quad \frac{8}{1} = \frac{\square}{24}$$

Students figure out the fraction that equals 1 inside the parentheses:

$$\frac{\text{bottles}}{\text{cartons}} \quad \frac{8}{1} \left( \frac{\square}{24} \right) = \frac{\square}{24}$$

The denominator of the fraction is 24, because  $1 \times 24 = 24$ . The top number of the fraction inside the parentheses must also be 24, because the fraction must equal 1:

$$\frac{24}{24} = 1.$$

Now the students multiply the numerators and solve the problem.

$$\frac{\text{bottles}}{\text{cartons}} \quad \frac{8}{1} \left( \frac{24}{24} \right) = \frac{192}{24} \quad \boxed{192 \text{ bottles}}$$

The same strategy works for division. For example:

There are 8 bottles in each carton.  
How many cartons hold 24 bottles?

$$\frac{\text{bottles}}{\text{cartons}} \quad \frac{8}{1} \left( \frac{3}{3} \right) = \frac{24}{3} \quad \boxed{3 \text{ cartons}}$$

Thus, as students develop confidence in solving problems, they also learn that there may be several methods that are equally successful.

The same ratio and proportion strategy may be applied to more advanced problems. For example:

3 cartons hold 12 bottles.  
How many bottles do 15 cartons hold?

Students set up a ratio equation as before:

$$\frac{\text{cartons}}{\text{bottles}} \quad \frac{3}{12} \left( \frac{5}{5} \right) = \frac{15}{60} \quad \boxed{60 \text{ bottles}}$$

## TABLES

The ability to organize and interpret data in table form is a valuable problem-solving skill. Consider the following problem:

Ann packed 69 paperback books and 51 hardback books in 2 boxes. One box is blue. The other box is red. She packed 62 books in the blue box, 15 of which are hardback.

- How many books are in the red box?
- Which box has more paperback books?
- How many hardback books are in the red box?
- What is the total number of books in both boxes?

Students learn how to map the data in a table, putting in the information the problem gives. In this type of table, the rows function as classification number families. For each row of the table, books is the big class, representing the big number. The types of books represent the small numbers:

<insert A here>

To complete the top row, students start with the big number (62) and subtract. To complete the bottom row, students add the two small numbers.

<insert B here>

The columns also function as number families. Students may now subtract in each of the three columns to complete the table:

<insert C here>

The answer to each of the four questions is found in the table.

- 58 books in the red box.
- More paperback books in the blue box.
- 36 hardback books in the red box.
- 120 books in both boxes.

## INVERSE OPERATION EQUATIONS

A variety of problem types may be solved by working backwards using inverse operations. Consider the following problems:

- Start with a number. Multiply by 3. Subtract 40. Then divide by 2. The answer is 7. What's the mystery number?*
- A charity must raise a certain amount of money through donations. The amount will be doubled by a matching fund. After paying bills of \$120 and \$58, the charity must have \$3,000 left for additional expenses. How much money must be raised through donations?*

These problem types are well suited for a work-backwards strategy, as the starting number is the unknown. The strategy applies inverse operations. The first step in the

strategy is to translate each event into an equation. For example, for the first problem students translate start with a number and multiply by 3 as:

$$\begin{array}{c} ? \\ \square \times 3 = \square \end{array}$$

The question mark denotes the mystery number. Neither the starting number nor the ending number is known. The ending number of the first equation becomes the starting number for the next equation. We subtract 40:

$$\begin{array}{c} ? \\ \square \times 3 = \square \\ \square - 40 = \square \end{array}$$

For the final equation we know that when we divide by 2 the answer is 7:

$$\begin{array}{c} ? \\ \square \times 3 = \square \\ \square - 40 = \square \\ \square \div 2 = 7 \end{array}$$

Now students can solve the problem by working backwards. They can figure out the starting number for each equation using inverse operations. They can “undo” division by multiplication, “undo” subtraction by addition, and so forth. First, they multiply by 2 for the last equation, which also gives them the ending number in the middle equation:

$$\begin{array}{c} ? \\ \square \times 3 = \square \\ \square - 40 = \boxed{14} \\ \boxed{14} \div 2 = 7 \end{array}$$

Next, add 40 for the middle equation:

$$\begin{array}{l} ? \\ \boxed{\phantom{00}} \times 3 = \boxed{54} \\ \boxed{54} - 40 = \boxed{14} \\ \boxed{14} \div 2 = 7 \end{array}$$

Finally, divide by 3 to figure out the mystery number:

$$\begin{array}{l} ? \\ \boxed{18} \times 3 = \boxed{54} \\ \boxed{54} - 40 = \boxed{14} \\ \boxed{14} \div 2 = 7 \end{array}$$

The answer is 18.

The same procedure may be applied to the second word problem, which asks about raising donations:

$$\begin{array}{l} ? \\ \boxed{\phantom{000}} \times 2 = \boxed{\phantom{000}} \\ \boxed{\phantom{000}} - 120 = \boxed{\phantom{000}} \\ \boxed{\phantom{000}} - 58 = 3000 \end{array}$$

Students work backwards to find the original amount:

$$\begin{array}{l} ? \\ \boxed{1589} \times 2 = \boxed{3178} \\ \boxed{3178} - 120 = \boxed{3058} \\ \boxed{3058} - 58 = 3000 \end{array}$$

The charity must raise \$1,589 through donations.

CONCLUSIONS

The NCTM sets forth an important vision that stands in stark contrast to the achievement level of most students with math disabilities, particularly with respect to word problems. This discrepancy will not be resolved unless we implement effective practices and curricula in our schools. Mapping represents a careful instructional approach that can help equip students with essential and powerful tools for solving word problems. Such instruction can help students with LD be successful in more advanced mathematical applications, enabling them to solve complex problems, reason mathematically, and make connections to the real world.

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$$\begin{array}{r}
 62 \\
 - 15 \\
 \hline
 47 \\
 \\
 69 \\
 + 51 \\
 \hline
 120
 \end{array}$$

	paperbacked	hardbacked	total books
blue	47	15	62
red			
both boxes	69	51	120

		paperbacked	hardbacked	total books
blue		15		62
red				
both boxes	69	51		

	paperbacked	hardbacked	total books
blue	47	15	62
red	22	36	58
both boxes	69	51	120